Spring 2017 MATH5012

Real Analysis II

Solution to Exercise 5

Standard notations are in force. * are for math-majors only. * are optional.

(1) Let $f \in L^1(\mathbb{R}^1)$ and $g \in L^p(\mathbb{R}),\ p \in [1, \infty]$.

(a) Show that Young's inequality also holds for $p = \infty$.

(b) Show that equality can hold in Young's inequality when $p = 1$ and ∞, and find the conditions under which this happens.

(c) For $p \in (1, \infty)$, show that equality in the inequality holds only when either f or g is zero almost everywhere.

(d) For $p \in [1, \infty]$, show that for each $\varepsilon > 0$, there exist $f \in L^1(\mathbb{R})$ and $g \in L^p(\mathbb{R})$ such that

$$\|f * g\|_p > (1 - \varepsilon)\|f\|_1 \|g\|_p \ .$$

Solution.

(a) It is obvious that fixing x, $f(x-y)g(y)$ is integrable w.r.t. y and

$$\begin{aligned} \left|\int f(x-y)g(y)dy\right| &= \int |f(x-y)g(y)|dy \\ &\leq \int |f(x-y)|dy\|g\|_\infty = \|f\|_1\|g\|_\infty < \infty, \forall x \in \mathbb{R}. \end{aligned}$$

Hence $\|f * g\|_\infty \leq \|f\|_1 \|g\|_\infty$

(b) For instance if g is constant function equals to 1 and f is nonnegative, then $\forall x \in \mathbb{R}$

$$\left|\int f(x-y)g(y)dy\right| = \left|\int f(x-y)dy\right| = \|f\|_1.$$

we see that $\|f*g\|_\infty = \|f\|_1$ and the Young's inequality holds for $p=\infty$. For $p=1$, if $f=g\geq 0$, then

$$\begin{aligned}\|f*f\|_1 &= \int\Big|\int f(y)f(x-y)dy\Big|dx = \int\int f(y)f(x-y)dydx \\ &= \int\int f(y)f(x-y)dxdy \\ &= \|f\|_1\int f(y)dy = \|f\|_1^2.\end{aligned}$$

(c) Suppose on the contrary, g and $f\neq 0$ a.e.. Since $0<\|f*g\|_p = \|f\|_1\|g\|_p<\infty$, the map

$$\Lambda(h) := \int h(x)F(x)dx$$

where $F(x)=f*g(x)$, is a well defined bounded linear map on $\mathcal{L}^q$. Moreover, substitute $h = sgn(F(x))|F(x)|^{\frac{p}{q}}$

$$\begin{aligned}\|f*g\|_p^p = \Lambda(h) &= \int h(x)F(x)dx = \int\int h(x)f(y)g(x-y)dydx \\ &= \int\int h(x)f(y)g(x-y)dxdy \\ &\leq \int |f(y)|\int |h(x)||g(x-y)|dxdy \\ &\leq \int |f(y)|\|h\|_q\|g\|_p dy \\ &= \|f\|_1\|g\|_p\|f*g\|_p^{\frac{p}{q}} \\ &= \|f*g\|_p^{\frac{p}{q}+1} = \|f*g\|_p^p\end{aligned}$$

Hence all the inequality hold. There is measurable A with $\mathcal{L}(A)>0$ such that $\forall y\in A, \int |h(x)||g(x-y)|dx = \|h\|_q\|g\|_p$. By the condition for Holder equality to hold (for example P.64-65 of Rudin's Real and

Complex Analysis), we see that there are $y_i \in A, i = 0, 1$ with $y_1 > y_0$

$$\frac{|F(x)|^p}{\|F\|_p^p} = \frac{|g(x - y_i)|^p}{\|g\|_p^p}, \forall x \in \mathbb{R} \setminus N_{y_i}$$

where N_{y_i} are some measure zero sets. We see that

$$|g(x - y_1)| = |g(x - y_0)|, \forall x \in \mathbb{R} \setminus \big(N_{y_0} \cup N_{y_1}\big).$$

Let $T := y_1 - y_0 > 0$, we have

$$|g(s + T)|^p = |g(s)|^p, \text{ a.e. } s \in \mathbb{R}$$

, which is absurd since $g \in \mathcal{L}^p$. Therefore f or $g = 0$ a.e..

(d) The case for $p = \infty$ follows from the example in b) which gives equality and nontrivial $f * g$. For $p \geq 1$, $\forall \varepsilon > 0$, let $f(x) = e^{-x}\chi_{[0.\infty)}(x)$, $g(x) = \chi_{[0.k)}(x)$, where k is to be chosen, then $\|f\|_1 = 1$ and $\|g\|_p = k^{\frac{1}{p}}$. It suffices to show that for sufficiently large k,

$$\int \Big(\int f(y)g(x - y)dy\Big)^p dx \geq (1 - \varepsilon)^p k.$$

In fact LHS,

$$\begin{aligned} \int \Big(\int f(y)g(x - y)dy\Big)^p dx &= \int_{x<0} + \int_{0\leq x\leq k} + \int_{k\leq x} \Big(\int f(y)g(x - y)dy\Big)^p dx \\ &=: I + II + III. \end{aligned}$$

It is immediate that $I = 0$. Moreover III is nonegative, we may estimate II. Since for $x \geq 0, -e^{-x} \geq -1$

$$II = \int_0^k \Big(\int_0^x f(y)dy\Big)^p dx = \int_0^k (1 - e^{-x})^p dx \geq \int_0^k (1 - pe^{-x}) dx \geq k - p$$

where we have used the Bernoulli's inequality. Hence

$$LHS \geq II \geq k - p \geq (1-\varepsilon)^p k$$

provided k is large enough.

(2) Show that for integrable f and g in $\mathbb{R}^n$,

$$\int f(x-y)g(y)\,dy = \int g(x-y)f(y)\,dy.$$

Solution.

Case 1. $f = \chi_E$ and $g = \chi_F$ for some measurable sets E and F.

$$\begin{aligned}\int f(x-y)g(y)\,dy = \int \chi_E(x-y)\chi_F(y)\,dy &= \int_{x-E} \chi_F \\ &= \mathcal{L}(F \cap (x-E)) = \mathcal{L}((F-x) \cap (-E)) \\ &= \mathcal{L}((x-F) \cap E) = \int_{x-F} \chi_E \\ &= \int \chi_F(x-y)\chi_E(y)\,dy = \int g(x-y)f(y)\,dy.\end{aligned}$$

Case 2. f, g are nonnegative measurable functions.

Pick sequences of increasing simple functions s_n and t_n such that $s_n \to f$ and $t_n \to g$. Then for each x, y, we have $s_n(x-y)t_n(y) \to f(x-y)g(y)$. By the Monotone Convergence Theorem,

$$\int f(x-y)g(y)\,dy = \int g(x-y)f(y)\,dy.$$

Case 3. f, g are integrable functions.

Consider f^+, f^-, g^+, g^- separately.

(3) A family $\{Q_\varepsilon\}$, $\varepsilon \in (0,1)$ or a sequence $\{Q_n\}_{n \geq 1}$ is called an "approximation

to identity" if (a) $Q_\varepsilon, Q_n \geq 0$, (b) $\int Q_\varepsilon, \int Q_n = 1$, and (c) $\forall \delta > 0$,

$$\int_{|x|\geq\delta} |Q_\varepsilon|(x)\,dx \to 0 \text{ as } \varepsilon \to 0 \text{ or}$$

$$\int_{|x|\geq\delta} |Q_n|(x)\,dx \to 0 \text{ as } n \to \infty.$$

Verify that

(i) $P_y(x) = \dfrac{1}{\pi}\dfrac{y}{x^2+y^2}$, $x \in \mathbb{R}$; $y \to 0$

(ii) $H_t(x) = \dfrac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}}$, $x \in \mathbb{R}^n$, $t \to 0$,

(iii) $\dfrac{1}{2\pi}F_k(x) = \begin{cases} \dfrac{1}{2\pi n}\dfrac{\sin^2\frac{kx}{2}}{\sin^2\frac{x}{2}}, & |x| \leq \pi, \\ 0, & |x| > \pi, \end{cases}$, $x \in \mathbb{R}$, $k \to \infty$

are approximations to identity.

Solution.

(i) A change of variable and the fact $\int \dfrac{1}{1+x^2}\,dx = \pi$ shows that

$$\pi \int P_y(x)\,dx = \int \frac{y}{x^2+y^2}\,dx = \pi.$$

A simple calculation shows that for every $\delta > 0$,

$$\int_{|x|\geq\delta} \frac{|y|}{x^2+y^2}\,dy = \pi + \arctan\left(\frac{-\delta}{|y|}\right) - \arctan\left(\frac{\delta}{|y|}\right)$$
$$\to \pi + \frac{-\pi}{2} - \frac{\pi}{2} = 0.$$

(ii) $\int H_t = 1$ follows from that $\int e^{-x^2}\,dx = 1$ and n iterations using Fubini's Theorem. Now for any $\delta > 0$, we claim that there exists an $1 > \varepsilon > 0$ such that whenever $0 < t < \varepsilon$, $H_t \leq H_1$ on the set $A = \{x \in \mathbb{R}^n : |x| \geq \delta\}$.

We choose an $\varepsilon \in (0,1)$ such that

$$0 < \frac{-2nt\log t}{1-t} \le \delta^2 \le |x|^2$$

whenever $0 < t < \varepsilon$. We can calculate that for these t,

$$H_t(x) \le H_1(x)$$

on A. Also, $H_t \to 0$ as $t \to 0$. By the Lebesgue Dominated Convergence Theorem, since $H_1 \in L^1(\mathbb{R}^n)$,

$$\lim_{t\to 0}\int_A H_t = \int_A \lim_{t\to 0} H_t = 0.$$

(iii) We first observe that the Fejer kernel, on $[-\pi,\pi]$,

$$F_k(x) = \frac{1}{k}\sum_{j=0}^{k-1} D_j(x) = \sum_{j=-k+1}^{k-1}\left(1-\frac{|j|}{k}\right)e^{ijx}$$

where $D_k(x) = \sum_{j=-k}^{k} e^{ijx}$. So $\int F_k = 1$. Fix $\delta > 0$. Then there exists a constant $c_\delta > 0$ such that for $|x| \ge \delta$, $\sin^2\frac{x}{2} \ge c_\delta$ and thus $|F_k(x)| \le \frac{1}{nc_\delta}$. It follows that $\int_{|x|\ge\delta} F_k = 0$.

(4) Let f be a continuous function in $\mathbb{R}^n$. Then $f * Q_\varepsilon \to f$ for any approximation to identity Q_ε.

Solution: Fix $x_0 \in \mathbb{R}^n$. Given any $\eta > 0$, there exists $\delta > 0$ such that

$$|f(x) - f(x_0)| < \eta$$

whenever $|x| < \delta$. Now

$$
\begin{aligned}
&|f * Q_\varepsilon(x_0) - f(x_0)| \\
&= \left| \int (f(x_0 - y) - f(x_0))\, Q_\varepsilon(y)\, dy \right| \\
&\leq \left| \int_{|y|<\delta} (f(x_0 - y) - f(x_0))\, Q_\varepsilon(y)\, dy \right| + \left| \int_{|y|\geq\delta} (f(x_0 - y) - f(x_0))\, Q_\varepsilon(y)\, dy \right| \\
&\leq \varepsilon \int_{|y|<\delta} |Q_\varepsilon(y)|\; dy + 2M \int_{|y|\geq\delta} |Q_\varepsilon(y)|\; dy
\end{aligned}
$$

where we take $M > 0$ such that $|f| \leq M$ (in order to have the integral $f * Q_\varepsilon$ defined, we need f to be integrable hence such M exists). It follows that

$$\overline{\lim}\, |f * Q_\varepsilon(x_0) - f(x_0)| \leq \varepsilon$$

because $\int_{|y|\geq\delta} Q_\varepsilon(y)\, dy \to 0$ implies $\int_{|y|<\delta} Q_\varepsilon(y)\, dy \to 1$.

(5) Improve (3) to: Let $f \in L^1(\mathbb{R}^n)$ and x a Lebesgue point of f. Then $f * Q_\varepsilon(x) \to f(x)$ as $\varepsilon \to 0$.

Solution. We focus on the special case where Q_ε is the standard mollifier

$$Q_\varepsilon(x) = \eta_\varepsilon(x) = \frac{1}{\varepsilon^n} \eta\left(\frac{x}{\varepsilon}\right),$$

and

$$\eta(x) = \begin{cases} c \exp\left(\dfrac{1}{|x|^2 - 1}\right), & \text{if } |x| < 1 \\ 0, & \text{if } |x| \geq 1 \end{cases}.$$

with

$$\int_{\mathbb{R}^n} \eta(x)\, dx = 1.$$

Now,

$$
\begin{aligned}
|f * \eta_\varepsilon(x) - f(x)| &\leq \frac{1}{\varepsilon^n} \int_{B_\varepsilon(x)} \eta\left(\frac{x-y}{\varepsilon}\right) |f(y) - f(x)| \; dy \\
&\leq |B_1| \, \|\eta\|_{L^\infty} \frac{1}{|B_\varepsilon|} \int_{B_\varepsilon(x)} |f - f(y)| \; dy \\
&\to 0 \quad \text{ as } \varepsilon \to 0.
\end{aligned}
$$